Advanced ODE-Lecture 12
Limit Set and Krasovskii's Theorem

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Outline

- **Motivation**
- **Limit Sets**
- **Krasovskii's Theorem**
- **Some Examples**
- **Summary**

Motivation

- How to understand an open orbit? Especially for its asymptotic behaviors! Limit set is one of best to characterize these asymptotic behaviors.

- Limit set is defined for better knowing asymptotic behavior of an open orbit. However, its essential and unexpected link to Lyapunov method has been found. So limit set goes beyond its original objective to find its extremely important application in advanced Lyapunov theory.

- Classical Lyapunov theory needs that Lyapunov function is positive definiteness and the derivative of Lyapunov function along any trajectory is negative definite. However, in most cases, like energy function as a Lyapunov function candidate, the derivative is only semi-positive definite. How to treat this situation? Krasovskii theorem comes to solve this issue.

Limit Set

Still consider a dynamic system

$$x' = f(x). \tag{12.1}$$

Definition 12.1 Let $x(t; x_0)$ be a solution of the system (12.1) for $t \in [0, \infty)$.

1) A point $x_0^0 \in D$ is an ω**-limit point** of $x(t; x_0)$ (or $\gamma(x_0)$) if there exists $\{t_n \geq 0\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} x(t_n; x_0) = x_0^0$. ω**-limit set** of $x(t; x_0)$ (or $\gamma(x_0)$) is denoted as $\Omega^+(x_0)$.

2) A point $x_0^0 \in D$ is an α**-limit point** of $x(t; x_0)$ (or $\gamma(x_0)$) if there exists $\{t_n \geq 0\}$ with $t_n \to -\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} x(t_n; x_0) = x_0^0$. The α**-limit set** of $x(t; x_0)$ (or $\gamma(x_0)$) is denoted as $\Omega^-(x_0)$.

Example 12.1 If $x(t;x_0) \equiv x_0$, then $\Omega^+(x_0) = \Omega^-(x_0) = \{x_0\}$. That is, the limit set of equilibrium is itself.

If $x(t;x_0)$ is a periodic orbit, then $\Omega^+(x_0) = \Omega^-(x_0) = x(t;x_0)$. That is, the limit set of a periodic orbit is also itself. (This is why an isolated closed orbit is called a limit cycle)

If an equilibrium $x = x_0$ is AS (unstable), then the equilibrium point is $\omega(\alpha)$ -limit set of its nearby trajectories; If $x(t;x_0)$ is stable (unstable) limit cycle, then $x(t;x_0)$ is $\omega(\alpha)$-limit set of its nearby trajectories.

Lemma 12.1 If $x = x^*$ is an ω-limit point of $x(t;x_0)$, then, any point on the trajectory $x(t;x^*)$ is also a ω-limit point of $x(t;x_0)$.

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Proof. Since $x = x^*$ is an ω-limit point, there exists $\{t_n \geq 0\}$ with $t_n \to \infty$ as $n \to \infty$ s. t. $\lim_{n\to\infty} x(t_n; x_0) = x^*$ by definition. Suppose that $x(\tau; x^*)$ is any point of $x(t; x^*)$. By Group Property (Lemma 11.3), we have

$$x(t_n + \tau; x_0) = x(\tau; x(t_n; x_0)).$$

Then

$$\lim_{n\to\infty} x(\tau + t_n;\ x_0) = \lim_{n\to\infty} x(\tau; x(t_n; x_0)) = x(\tau; \lim_{n\to\infty} x(t_n; x_0)) = x(\tau; x^*).$$

This shows that $x(\tau; x^*)$ is also an ω-limit point.

Remark 12.1 Lemma 12.1 shows that $\Omega^+(x_0)$ consists of whole trajectories of (12.1). This is a very important property for a dynamic system. It is not true for time-varying systems in general. Why? See the proof of Lemma 12.1.

Lemma 12.2

The $\omega(\alpha)$-limit set $\Omega^+(x_0)(\Omega^-(x_0))$ of a trajectory $\gamma(x_0)$ is closed subset of D;

Proof. First, it follows $\Omega^+(x_0) \subset D$ by Lemma 12.1. To show that $\Omega^+(x_0)$ is closed in D, let $\{p_n\} \in \Omega^+(x_0)$ with $\lim_{n\to\infty} p_n = p$ and show $p \in \Omega^+(x_0)$. Since $p_n \in \Omega^+(x_0)$, then, for each fixed $n \geq 1$, $\exists$ $\{t_k^{(n)}\}$ with $t_k^{(n)} \to \infty$ as $k \to \infty$ s.t.

$$\lim_{k\to\infty} x(t_k^{(n)}; x_0) = p_n .$$

Moreover, we may assume that $t_k^{(n+1)} > t_k^{(n)}$ since otherwise we may choose a subsequence of $t_k^{(n)}$ with this increasing property. Then, $\lim_{k\to\infty} x(t_k^{(n)}; x_0) = p_n$ implies that for each n, $\exists$ sufficiently large $T_n > 0$ s.t. for $k > T_n$, one has

$$\| x(t_k^{(n)}; x_0) - p_n \| < \frac{1}{n}.$$

Let $t_n = t_{T_n}^{(n)}$, depending on n. Then, $t_n \to \infty$ and by the triangle inequality,

$$\| x(t_n; x_0) - p \| \le \| x(t_n; x_0) - p_n \| + \| p_n - p \| \le \frac{1}{n} + \| p_n - p \| \to 0$$

as $n \to \infty$. Thus, $p \in \Omega^+(x_0)$. Hence, $\Omega^+(x_0)$ is a closed subset of D. □

Theorem 12.1 If $\gamma(x_0) \subset D$ is bounded for all $x_0 \in D$, then $\Omega^+(x_0)\,(\Omega^-(x_0))$ is

1) nonempty;
2) compact;
3) invariant;
4) connected subset of D;
5) $x(t; x_0) \to \Omega^+(x_0)$ as $t \to \infty$ ($x(t; x_0) \to \Omega^-(x_0)$ as $t \to -\infty$).

Proof. 1) Since $\gamma(x_0)$ is bounded, there exists a convergent subsequence $\{x(t_n; x_0) \subset \gamma(x_0)\}$ such that $x(t_n; x_0) \to p \in \Omega^+(x_0)$ as $t_n \to \infty$ by the Bolzano-Weierstrass theorem. So $\Omega^+(x_0)$ is nonempty.

2) For any $p_n \in \Omega^+(x_0)$ with $p_n \to p$ as $n \to \infty$, we will show that $p \in \Omega^+(x_0)$.

Since $p_n \to p$ as $n \to \infty$, for $\forall \varepsilon > 0$, $\exists$ $n_0 > 0$ such that

$$\| p_{n_0} - p \| \le \frac{\varepsilon}{2}.$$

But $p_{n_0} \in \Omega^+(x_0)$, there exists $\{t_n \ge 0\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\| x(t_n; x_0) \to p_{n_0} \| \le \frac{\varepsilon}{2}, \quad n >> 1.$$

Then,

$$\| x(t_n; x_0) - p \| \le \| x(t_n; x_0) - p_{n_0} \| + \| p_{n_0} - p \| \le \varepsilon, \quad n >> 1.$$

So $p \in \Omega^+(x_0)$. Then, $\Omega^+(x_0)$ is closed. Since $\Omega^+(x_0)$ is also bounded by assumption. Therefore, $\Omega^+(x_0)$ is compact.

3) Let $p \in \Omega^+(x_0)$ and show that $x(t;p) \in \Omega^+(x_0)$ for all $t \geq 0$. Since, $p \in \Omega^+(x_0)$, $\exists$ $\{t_n \geq 0\}$ with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n; x_0) \to p$ as $n \to \infty$. By the Group property,

$$x(t + t_n; x_0) = x(t; x(t_n; x_0)),$$

where for sufficiently large $n > 0$, $t + t_n \geq 0$. By the continuity,

$$\lim_{n \to \infty} x(t + t_n; x_0) = \lim_{n \to \infty} x(t; x(t_n; x_0)) = x(t; p),$$

which shows that $x(t;p) \in \Omega^+(x_0)$ for all $t \geq 0$. Therefore, $\Omega^+(x_0)$ is invariant with respect to the flow x_t of (12.1).

4) Suppose that $\Omega^+(x_0)$ is not connected by contradiction. Then, there exist two non-empty, disjoint, closed sets A and B such that $\Omega^+(x_0) = A \cup B$.

Let $d = \inf_{x \in A,\, y \in B} \| x - y \| > 0$. Since the points of A and B are ω-limit point of $\Omega^+(x_0)$, for $d > 0$, there exists arbitrarily large $t > 0$ such that

$$d(x(t; x_0), A) < \frac{d}{2} \text{ and } d(x(t; x_0), B) < \frac{d}{2}.$$

Let $g(t) = d(x(t; x_0), A)$. Since $g(t)$ is a continuous function, then, ∃ $t_n \to \infty$ s.t.

$$d(x(t_n; x_0), A) = \frac{d}{2} \text{ for all } n \geq 0.$$

Since $x(t_n; x_0) \subset K$, ∃ a subsequence converging to $p \in \Omega^+(x_0)$. So it follows that $d(p, A) = \frac{d}{2}$. But $d(p, B) \geq d(A, B) - d(p, A) = d - \frac{d}{2} = \frac{d}{2}$, which implies that $p \notin A$ and $p \notin B$. Then, $p \notin \Omega^+(x_0)$. This is a contradiction. Thus, $\Omega^+(x_0)$ is connected. Similar to show that $\Omega^-(x_0)$ is connected.

5) suppose that $x(t;x_0) \to \Omega^+(x_0)$ as $t \to \infty$ is not true. Then, $\exists$ $\varepsilon > 0$ and $\{t_n \geq 0\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$d(x(t_n;x_0), \Omega^+(x_0)) > \varepsilon .$$

Since $x(t_n;x_0) \subset K$, there exists a convergent subsequence $x(t_{n_k};x_0) \to p$ as $k \to \infty$. Then, $p \in \Omega^+(x_0)$ and at the same time, it keeps that $d(p, \Omega^+(x_0)) > \varepsilon$.

This is a contradiction. It is similar to show for $\Omega^-(x_0)$.

Krasovskii's Theorem

Consider the pendulum equation with friction given by

$$\begin{cases} x_1' = x_2 \\ x_2' = -\dfrac{g}{l}\sin x_1 - \dfrac{k}{m}x_2 \end{cases}. \qquad (12.2)$$

One has $V(x) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x_2^2 > 0\ (-2\pi < x_1 < 2\pi) \Rightarrow V'(x) = -\frac{k}{m}x_2^2 \le 0$.

We see that $V'(x) < 0$ except for $x_2 = 0$, where $V'(x) = 0$. For the system (12.2) maintaining $V'(x) = 0$, the trajectory must be confined to $x_2 = 0$. Unless $x_1 = 0$, it is impossible from the pendulum equation with friction

$$x_2(t) \equiv 0 \Rightarrow x_2'(t) \equiv 0 \Rightarrow \sin x_1(t) \equiv 0 \Rightarrow x_1(t) \equiv 0.$$

Therefore, $V(x(t))$ must decrease toward to zero and, consequently, $\lim_{t\to\infty} x(t) = 0$ because $V(x)$ is positive definite. This is consistent with the fact that, due to friction, energy can't remain constant while the system is in motion.

Remark 12.2 If no trajectories staying identically at points where $V'(x) = 0$ are assumed, except at the origin, the origin is AS in the case of $V(x) > 0$ and $V'(x) \le 0$. This is a basic idea of Krasovskii's Theorem, which links to limit set.

Theorem 12.2 (Krasovskii's Theorem) Let $V: D \to R$ be C^1, such that

$$V(0)=0 \text{ and } V(x)>0 \text{ in } D-\{0\};$$

$$V'(x) \le 0 \text{ in } D.$$

Let $S=\{x \in R^n \mid V'(x)=0\}$. If there is no solution can stay identically in S, other than the trivial solution (origin). Then, the origin of (12.1) is AS.

Proof. By the given Lyapunov conditions, the origin is stable. We only need to show its attraction. That is, $\lim_{t \to +\infty} x(t; x_0) = 0$. If we can show that $\Omega(x_0)=\{0\}$, then we can get $\lim_{t \to +\infty} x(t; x_0) = 0$.

First, we can find $\Omega_\eta \subset B_r \subseteq D$ such that Ω_η is invariant by Theorem 9.1. For any $x_0 \in \Omega_\eta$, we have $x(t; x_0) \subset \Omega_\eta$ for all $t \geq 0$. Therefore, $\Omega(x_0) \subseteq \Omega_\eta$ is non-empty because of Lemma 12.1.

Next, we show that $\Omega(x_0) = \{0\}$. By contradiction, if $\exists$ $\{t_n \geq 0\}$ with $t_n \to \infty$ as $n \to \infty$ s.t. $\lim_{n\to\infty} x(t_n; x_0) = x^* \neq 0$. By the given Lyapunov conditions, we know that $\lim_{n\to\infty} V(x(t_n; x_0))$ exists. Then, we have

$$\lim_{n\to\infty} V(x(t_n; x_0)) = V(x^*) > 0. \tag{12.3}$$

For $x(t; x^*)$, by the second Lyapunov condition, we have

$$V(x(t; x^*)) \leq V(x^*).$$

If $V(x(t;x^*)) \equiv V(x^*)$ for all $t \geq 0$, then $V'(x(t;x^*)) \equiv 0$, which shows that $x(t;x^*) \subseteq S$, for all $t \geq 0$. This is a contradiction to the assumption of the theorem.

Then, $\exists$ $\tau > 0$ s.t. $V(x(\tau;x^*)) < V(x^*)$. By Lemma 12.1, $x(\tau;x^*)$ is also a ω-limit point of $x(t;x_0)$. $\exists$ $\{\bar{t}_n\}$ with $\bar{t}_n \to \infty$ as $n \to \infty$ s. t.

$$\lim_{n\to\infty} x(\bar{t}_n;\ x_0) = x(\tau;x^*) .$$

Therefore, we have

$$\lim_{n\to\infty} V(x(\bar{t}_n;\ x_0)) = V(x(\tau;x^*)) < V(x^*) .$$

This also contradicts to (12.3). This contradiction shows that $\Omega(x_0) = \{0\}$. Therefore,

$$\overline{\lim_{t\to+\infty}}\, x(t;x_0) = \underline{\lim_{t\to+\infty}}\, x(t;x_0) = 0 \quad \Rightarrow \quad \lim_{t\to+\infty} x(t;x_0) = 0 . \quad \square$$

Remark 12.3 Krasovskii's Theorem suggests that there exists some links among Lyapunov stability, invariance property and ω-limit set. This insight is well developed by LaSalle. So Krasovskii's Theorem is also called as LaSalle-Krasovskii's Theorem in books. The more general case of this theorem is called LaSalle's Invariance principle, which will be stated next class.

Theorem 12.3 (Krasovskii's Theorem for global) Let $V: D \to R$ be a continuously differentiable function, such that the following Lyapunov conditions are satisfied

$$V(0)=0 \text{ and } V(x)>0 \text{ in } D-\{0\};$$

$$V'(x) \le 0 \text{ in } D;$$

$$\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty.$$

Let $S=\{x \in R^n \mid V'(x)=0\}$. If there is no trajectory can stay identically in S, other than the origin. Then, the origin of (12.1) is GAS. **Proof. (Homework).**

Example 12.2 Consider the general pendulum equation

$$\begin{cases} x_1' = x_2 \\ x_2' = -g(x_1) - h(x_2) \end{cases}$$

where $g(\cdot)$ and $h(\cdot)$ are locally Lip. and satisfy

$$g(0) = 0, \quad yg(y) > 0, \quad \forall\, y \neq 0, \quad y \in (-a, a);$$

$$h(0) = 0, \quad yh(y) > 0, \quad \forall\, y \neq 0, \quad y \in (-a, a).$$

If we take

$$V(x) = \int_0^{x_1} g(y)\,dy + \frac{1}{2}x_2^2 .$$

on $D = \{x \in R^2 \mid -a < x_j < a\}$, then $V(x)$ is positive definite in D.

$$V'(x) = g(x_1)x_2 + x_2[-g(x_1) - h(x_2)] = -x_2 h(x_2) \leq 0 .$$

Then, $S = \{x \in D \mid V'(x) = 0\}$, note that

$$V'(x) = 0 \quad \Rightarrow \quad x_2 h(x_2) = 0 \quad \Rightarrow \quad x_2 = 0, \text{ since } -a < x_2 < a .$$

Hence, $S=\{x\in D \mid x_2=0\}$. If $x(t)$ belongs identically to S. Then,

$$x_2(t)\equiv 0 \;\Rightarrow\; x_2'(t)\equiv 0 \;\Rightarrow\; g(x_1(t))\equiv 0 \;\Rightarrow\; x_1(t)\equiv 0.$$

Therefore, the trajectory that can stay identically in S is only $x(t)\equiv 0$. Thus, the origin is AS.

Example 12.3 In Example 12.2 with $a=\infty$ and $g(\cdot)$ satisfies

$$\int_0^y g(z)dz \to \infty \quad \text{as} \quad |y| \to \infty.$$

The Lyapunov function

$$V(x)=\int_0^{x_1} g(y)dy+\frac{1}{2}x_2^2$$

is radially unbounded. Similar to the previous example, it can be shown that $\dot{V}(x)\le 0$ in R^2, and

$$S=\{x\in R^2 \mid V'(x)=0\}=\{x\in R^2 \mid x_2=0\}$$

contains no trajectory other than $x(t)\equiv 0$. Hence, the origin is GAS.

Summary

- Krosovskii's Theorem is a fundamental result in advanced Lyapunov theory. Any possible extensions to time-varying systems, hybrid systems, large scales systems etc. are still playing a key role and profound influence in system analysis. So its development will be interested by researchers.

- Krasovskii's Theorem uses an additional condition to treat the case of $V(x) > 0$ and $V'(x) \leq 0$. This type of $V(x)$ is called a non-strict Lyapunov function. Many nonlinear systems have such a Lyapunov function. However, in controller design, a strict Lyapunov function is preferred, i.e. $V(x) > 0$ and $V'(x) < 0$. The construction of the strict Lyapunov function based on a known non-strict Lyapunov function plays a central role in nonlinear control. It receives recently much attention. Especially it is for time-varying systems. The corresponding theory is said "Strictification Method". The reference book is "Construction of Strict Lyapunov Functions" by Michael Malisoff and Frédéric Mazenc, Published by Springer, in 2009. However, it is still lots of questions unsolved and open.

Homework

Prove Theorem 12.3